Recitation 5. April 6

Focus: linear transformations, change of basis, determinants

A **linear transformation** is a function $\phi : \mathbb{R}^n \to \mathbb{R}^m$ such that for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have:

$$\boxed{\phi(\boldsymbol{v}+\boldsymbol{w}) = \phi(\boldsymbol{v}) + \phi(\boldsymbol{w})} \qquad \text{and} \qquad \boxed{\phi(\alpha\boldsymbol{v}) = \alpha\phi(\boldsymbol{v})}$$

A linear transformation ϕ can be expressed as a matrix B, with respect to given bases $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ of $\mathbb{R}^n$ and $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$ of $\mathbb{R}^m$: the entry b_{ij} on the i–th row and j–th column of B are such that:

$$\boxed{\phi(x_1\boldsymbol{v}_1 + ... + x_n\boldsymbol{v}_n) = (b_{11}x_1 + ... + b_{1n}x_n)\boldsymbol{w}_1 + ...(b_{m1}x_1 + ... + b_{mn}x_n)\boldsymbol{w}_m}$$

Changing the bases $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ and $\boldsymbol{w}_1, ..., \boldsymbol{w}_m$ will mean different coefficients b_{ij}, and hence a different matrix B, for one and the same function ϕ. The general rule is the **change of basis** formula:

$$\boxed{B = W^{-1}AV}$$

where $V = \left[\ \boldsymbol{v}_1 \mid \dots \mid \boldsymbol{v}_n\ \right]$, $W = \left[\ \boldsymbol{w}_1 \mid \dots \mid \boldsymbol{w}_n\ \right]$, and A is the matrix which represents ϕ in the standard basis:

$$\boxed{\phi(\boldsymbol{v}) = A\boldsymbol{v}} \quad \Leftrightarrow \quad \boxed{\phi(x_1\boldsymbol{e}_1 + ... + x_n\boldsymbol{e}_n) = (a_{11}x_1 + ... + a_{1n}x_n)\boldsymbol{e}_1 + ...(a_{m1}x_1 + ... + a_{mn}x_n)\boldsymbol{e}_m}$$

We note that if $\phi(\boldsymbol{v}) = A\boldsymbol{v}$ and $\psi(\boldsymbol{v}) = B\boldsymbol{v}$, then $\phi \circ \psi(\boldsymbol{v}) = (AB)\boldsymbol{v}$. Moreover, $\phi^{-1}(\boldsymbol{v}) = A^{-1}\boldsymbol{v}$, assuming the linear transformation ϕ has an inverse, which is equivalent to A being invertible.

Given a square matrix A, its **determinant** (denoted by $\det A$) is the factor by which the linear transformation $\phi(\boldsymbol{v}) = A\boldsymbol{v}$ scales volumes of regions in $\mathbb{R}^n$. It satisfies the property that:

$$\boxed{\det(AB) = (\det A)(\det B)}$$

A computationally efficient way to compute the determinant is to put A in row echelon form, and set:

$$\boxed{\det A = (-1)^{\#}(\text{product of pivots})}$$

where # is the number of row exchanges that you need to do as you put A in row echelon form. Note the identities:

$$\boxed{\det A^T = \det A} \qquad \boxed{\det A^{-1} = \frac{1}{\det A}} \qquad \boxed{\det(\lambda A) = \lambda^n \det A}$$

for an $n \times n$ matrix A.

1. Recall that the linear transformation "counter-clockwise rotation by an angle α" is represented in the standard basis of $\mathbb{R}^2$ by the matrix:

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

If you compose a rotation by angle α with a rotation by angle β, what do you get geometrically? What is the matrix that represents this composition? Can you use this to get formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$?

Solution:

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2. Determine whether the following maps $\phi_a, \phi_b, \phi_c : \mathbb{R}^3 \to \mathbb{R}^3$ are linear. If so, find a matrix representation of the map in terms of the standard basis of $\mathbb{R}^3$, and then find a matrix representation in terms of the basis:

$$\boldsymbol{v}_1 = \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \boldsymbol{w}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \boldsymbol{w}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(a) $\phi_a \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y + z \\ x^2 + y^2 + z^2 \\ 0 \end{bmatrix}.$

(b) $\phi_b(\boldsymbol{v}) = (\boldsymbol{a} \cdot \boldsymbol{v})\boldsymbol{a}$, where $\boldsymbol{a} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \in \mathbb{R}^3.$

(c) $\phi_c \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x - y - z \\ x + 2y \\ y - 3z \end{bmatrix}.$

Solution:

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3. Compute the determinant of:

$$\begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & -4 & -2 \\ 1 & 3 & -1 & 2 \\ -1 & 3 & 0 & 5 \end{bmatrix}$$

by using row operations.

Solution: